Utility Maximization

Theory: A person's happiness, or *total utility*, depends upon the decisions that she makes throughout her life.

Important decisions include: how much to work, how much to save, what/how much to buy.

<u>Assumptions often made about utility</u>: although not necessary, it is often realistic to make the following assumptions concerning the behavior of a person's utility:

1) Non-satiation: Higher consumption of products that people buy, holding all else equal, leads to higher total utility.

2) Logic: A person is able to *rank* the utility derived from of all possible decisions that she faces (from highest to lowest). This requires that *completeness of preferences*; the individual is aware of all options available to her. It also requires *transitivity*; if choice B is better than choice A, and choice C is better than choice B, then it better be true that choice C is better than choice A.

3) Diminishing Marginal Utility: *Marginal utility* measures the increase in total utility attained when an activity (such as consumption of a good) is incrementally increased. If you prefer,

$$\text{marginal utility} = \text{increase in total utility} \div \text{increase in activity}$$

Diminishing marginal utility means, for example, that eating the 2^{nd} potato chip causes a larger increase in utility than eating the 3rd potato chip. (Note, however, that the 3rd potato chip does not reduce total utility; it causes total utility to rise by less than the 2^{nd} chip.)

Example: (below, we measure utility in units known as *utils*)

Potato chips eaten	Marginal chip utility	Total chip utility
0	--	0
1	50	50
2	40	90
3	30	120
4	15	135
5	5	140

Algebraic Representation of Utility:

U = f(decisions)

"U" is a number of utils representing the happiness of the individual. The higher the value of U, the better off is the individual.

Often to maintain tractability (and to be able to graph the individual's behavior) we sometimes limit an individual's choice to this: the individual may buy only two types of goods; how many units of each should she buy? Given this limitation, there are many different functional forms that we could use to represent an individual's utility. Many realistic forms are quite complex mathematically and we shall omit those. Below are three less realistic, but more tractable examples of utility functions: the Cobb-Douglas utility function, the linear utility function, and the Leontief utility function.

Example 1: The *Cobb-Douglas* utility function represents the utility possible from consuming two goods, x and y. In general, a Cobb-Douglas function has the form

$U = X^aY^{(1-a)}$

where X is units of x consumed
Y is units of y consumed
U is "utils" of total utility
a is a constant less than 1

Here's a specific example of a Cobb-Douglas utility function, for an individual who consumes only beer and pizza:

$U = \text{units of beer consumed}^{.4} \times \text{units of pizza consumed}^{.6}$

Example 2: The *Linear* utility function is another (sometimes not too realistic) representation of the utility gained by consuming goods x and y.

$U = aX + bY$

where X is units of x consumed
Y is units of y consumed
U is "utils" of total utility
a and b are positive constants

Here's a specific example of a linear utility function, for an individual who consumes only Coke and Pepsi:

U = (10 x units of Coke consumed) + (10 x units of Pepsi consumed)

Example 3: The *Leontief* utility function is another (sometimes not too realistic) representation of the utility gained by consuming goods x and y.

U = min[aX, bY]

where X is units of x consumed
Y is units of y consumed
U is "utils" of total utility
a and b are positive constants

Here's a specific example of a Leontief utility function, for a (pantsless) individual who consumes only Shirts and Buttons:

U = minimum of [1 x # of shirts consumed, 1/6 x # of buttons consumed]

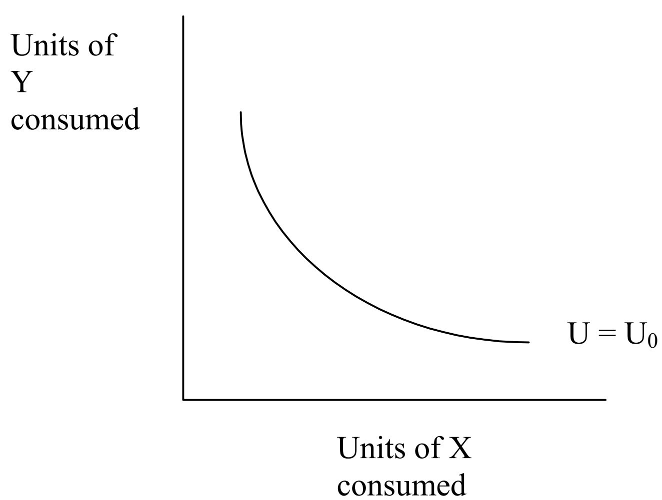
Total Utility Graphed Using Indifference Curves

One can use indifference curves to graph a person's prospective total utility levels

An *indifference curve* represents all combinations of two goods x and y that provide the individual with an equal level of total utility.

(A related concept is the *marginal rate of substitution (MRS)*: the amount of X that one must be given, to compensate for a loss of good Y, in order to maintain total utility at a constant level.)

Example 1: If one draws an indifference curve for a Cobb-Douglas utility function then it will look as this:



Note this about the above indifference curve:

--the negative slope: This indicates that the if one takes some X from the consumer, then she must be given more Y in order to maintain her total utility at its initial level.

--the convex curve: This indicates diminishing MRS

Example 1 continued:

Here are 4 combinations of beer and pizza that provide 100 utils of utility to an individual with this utility function: (numbers are rounded)

$U = B^{.4} Z^{.6}$

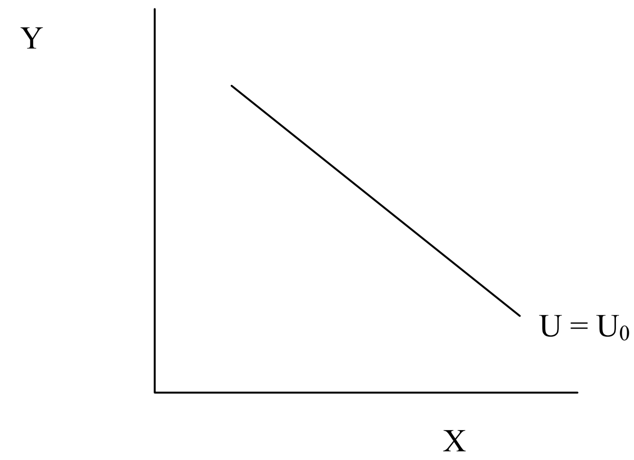
point	**Units of beer, B**	**Units of pizza, Z**	**Total utility, U ($=B^{.4}Z^{.6}$)**
A	100	100	100 utils
B	90	107.3	100 utils
C	80	116	100 utils
D	70	126.8	100 utils

If you were to graph point A B C and D and connect them, you would have a part of this dude's indifference curve representing U=100 utils of utility. The entire U=100 utils indifference curve constitutes ALL combinations of beer and pizza that provide 100 utils of utility. It would have a convex curvy shape similar to the indifference curve on page 3.

Note the MRS between points A and B: If one takes **10 units of beer** from this dude (reducing him from 100 units of beer to 90 units of beer), then one must compensate him with an extra **7.3 units of pizza** (increasing him from 100 to 107.3) in order to maintain his utility at 100 utils. Hence his MRS between A and B = 10 beer / 7.3 pizza = 1.3699

Verify that MRS between B and C = 1.1494
Verify that MRS between C and D = .9259
See how the MRS diminishes as one moves along the indifference curve?

Example 2: An indifference curve for a linear utility function



Note:

--the negative slope: This indicates that the if one takes some X from the consumer, then she must be given more Y in order to maintain her total utility at its initial level.

--the lack of a curve: x and y are perfect substitutes.

Example 2 continued:

Here are 4 combinations of coke and pepsi that provide 100 utils of utility to an individual with this utility function:

$$U = 10C + 10E$$

point	**Units of coke, C**	**Units of pepsi, E**	**Total utility, U (=10C + 10E)**
A	10	0	100 utils
B	9	1	100 utils
C	8	2	100 utils
D	7	3	100 utils

If you were to graph point A B C and D and connect them, you would have a part of this dude's indifference curve representing U=100 utils of utility. The entire U=100 utils indifference curve constitutes ALL combinations of coke and pepsi that provide 100 utils of utility. It would have a linear shape similar to the indifference curve on this page .

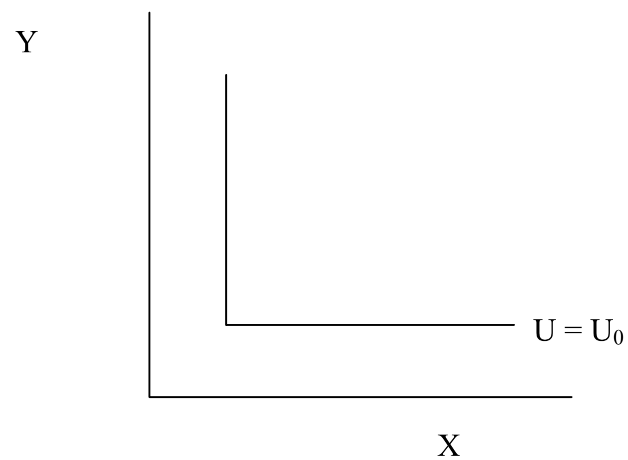
Note the MRS between points A and B: If one takes **1 units of coke** from this dude (reducing him from 10 units of beer to 9 units of coke), then one must compensate him with an extra **1 units of pepsi** (increasing him from 0 to 1) in order to maintain his utility at 100 utils. Hence his MRS between A and B = 1 coke / 1 pepsi = 1

Verify that MRS between B and C = 1

Verify that MRS between C and D = 1

See how the MRS is constant?

Example 3: An indifference curve for a Leontief utility function



Note:

--the goofy shape: This indicates that the if one gives some extra X to the consumer without giving her any more Y, then her total utility does not rise! (X may be "left shoes," and Y may be "right shoes.") Hence X and Y are perfect complements.

Example 3 continued:

Here are 4 combinations of shirts and buttons that provide 100 utils of utility to an individual with this utility function:

U = minimum of [1 x # of shirts consumed, 1/6 x # of buttons consumed]

or U = min[S, .166667B]

point	**Units of shirts,S**	**Units of buttons,B**	**Total utility,U**
A	100	600	100 utils
B	100	1 million	100 utils
C	100	1 billion	100 utils
D	1 trillion	600	100 utils

If you were to graph point A B C and D and connect them, you would have a part of this dude's indifference curve representing U=100 utils of utility. The entire U=100 utils indifference curve constitutes ALL combinations of coke and pepsi that provide 100 utils of utility. It would have an L shape similar to the indifference curve on page 6.

Note that the MRS in this instance is undefined.

Utility maximization and indifference curves: For any utility function, there are actually an infinite number of indifference curves—one for each possible utility level. It is generally desirable for the individual to "get on" the highest indifference curve that she is able to; this indicates that she is maximizing her total utility.

The indifference curve and the MRS: Notice in examples 1 and 2 that we calculated the MRS for segments of an indifference curve between two points on the curve. Roughly speaking the MRS was the "**slope**" of the indifference curve between the two points.

But suppose we want to calculate the MRS for an infinitesimally small segment of the indifference curve; that is, we want the MRS at one single point on the indifference curve? Well, take any point on an indifference curve. *The MRS is the (absolute value of) the **slope** of a line tangent to the indifference curve at that point.*

Calculating Marginal utility if one has a specific utility function

Recall that marginal utility measures the increase in a person's total utility if he consumes one more unit of a good. Let's calculate the marginal utility of each of the first 3 pizzas for this dude, assuming that he has 100 units of beer:

(Note that in order to calculate marginal utility for a good, one must give the individual a certain amount of the other good, then hold the amount of that other good constant.)

$$U = B^{.4} Z^{.6}$$

Units of beer, B	Units of pizza, Z	Total utility, $U = B^{.4}Z^{.6}$	Marginal utility
100	0	0	Not defined
100	1	6.309573	6.309573
100	2	9.563525	3.253952
100	3	12.19755	2.634029

<u>The MRS and marginal utility (or MU):</u> The ratio of the marginal utilities of X and Y equals the marginal rate of substitution:

$MRS = MU_x/MU_y$

(To read the above equation, say "the marginal rate of substitution is the marginal utility of good X divided by the marginal utility of good Y.")

This makes sense, since the number of units of X that one must receive, after a loss in Y, in order to maintain one's utility, depends upon the relative utility gained/lost from an exchange of Y for X.

For example,
Suppose the marginal utility of X is 10 utils, and the marginal utility of Y is 20 utils. Well, if you take 1 unit of Y from the dude (suffering the dude with a loss of 20 utils), you must give him 2 units of X (each worth 10 utils) to maintain his total utility at its original level.

Hence the MRS = 1 unit of Y / 2 units of X = **.5**

Note that MU_x/MU_y = **.5** also!!! Coincidence? No!!!!!

(Did you really follow the last bit of stuff? In you did not, then don't just skip to the next section! Go back over the last section again and again until you get it! If you don't get it after that then give me a call or come see me in my office!)

Constraints facing the individual

No individual can attain an infinitely high level of utility. This is because each individual faces some barriers, or constraints, that limit her feasible choices. Generally one faces *time* constraints and income *constraints*. In addition, one must usually buy the things that one consumes.

We shall often ignore (explicitly, anyway) the time constraint and consider only the income constraint.

Income constraint: general example
Consider our individual who can buy two goods, x and y, at price p_x per unit of x and p_y per unit of y. Suppose that this individual has income I.

This individual's budget constraint can be written

$$p_xX + p_yY = I$$

For ease of graphing, we can solve this equation for Y:

$$Y = I/p_y - (p_x/p_y)X$$

Hence an income constraint is a negatively-sloped line with vertical intercept I/p_x and slope $-(p_x/p_y)$:

Income constraint: specific example
Consider Bozo, who can buy two goods, cheese and donuts, at price $5 per unit of cheese (C) and $2 per unit of donuts (D). Suppose that Bozo has $100 of income
.

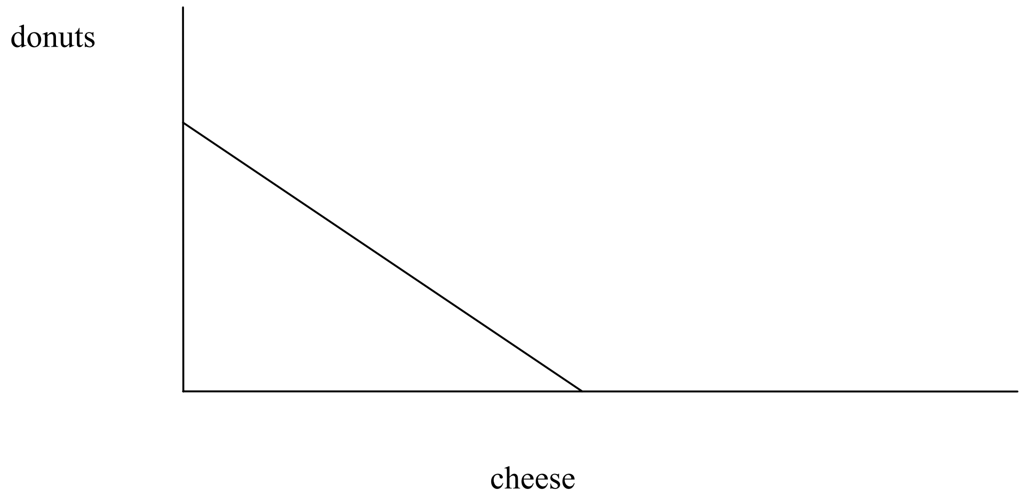
Bozo's budget constraint can be written

$$5C + 2D = 100$$

For ease of graphing, we can solve this equation for units of donuts:

$$D = 50 - 2.5C$$

Hence Bozo's income constraint is a negatively-sloped line with vertical intercept 50 and slope –2.5:



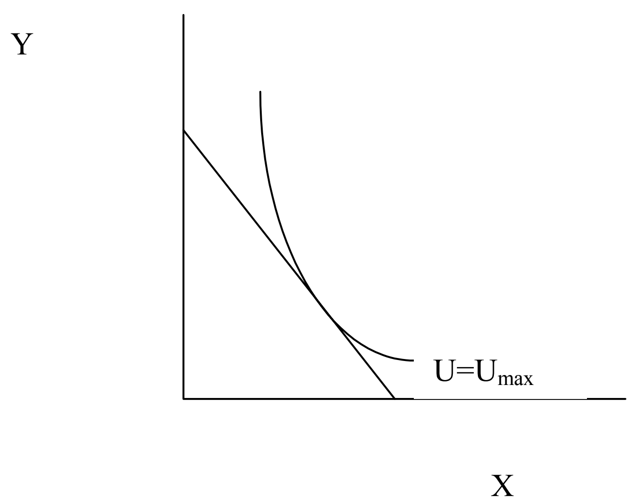
When graphed, an income constraint is sometimes called a budget line.

Interpretation of the budget line. With her limited income, the individual can only afford to consume combinations of x and y *on or inside* the budget line. (By "inside the budget constraint," I mean between it and the origin.) This is sometimes called the *feasible area*.

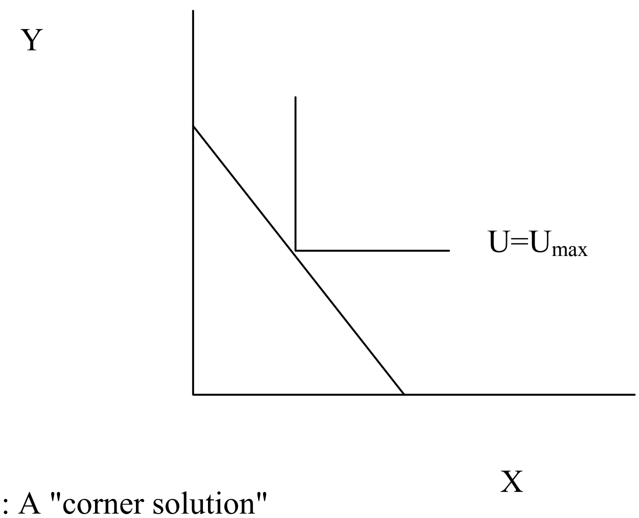
Utility Maximization Graphed

The highest indifference curve that one can "get on" must have at least one of its points in the feasible area. Usually, in fact, it only has 1 point in the feasible area. Here are some examples for a consumer who can buy goods X and Y.

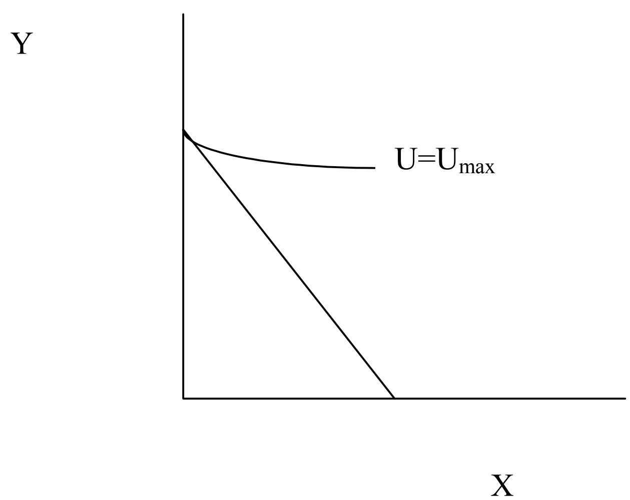
Example 1 (the most realistic): Cobb-Douglas



Example 2: Leontief



Example 3: A "corner solution"



An algebraic utility-maximizing rule:

If one has a nicely-behaved utility function, such as a Cobb-Douglas, then at the utility-maximizing point there will be a tangency between the indifference curve and the budget line. In other words, utility-maximization requires

MRS = slope of budget line

or

$MU_x/MU_y = p_x/p_y$

rewritten

$MU_x/p_x = MU_y/p_y$

In this last form, this can be interpreted as: the marginal utility *per dollar* of X equals the marginal utility *per dollar* of Y.

The calculus of utility maximization: a general example

Consider an individual with Cobb-Douglas utility function

$U = X^aY^{(1-a)}$

and budget constraint

$I = p_xX + p_yY$

We can set up a constrained maximization problem to calculate how many units of X and Y that this person will buy.

maximize $U = X^aY^{(1-a)}$

subject to: $I = p_xX + p_yY$

Solution of this problem, using the Lagrangean method, is undertaken in the textbook, so I will avoid reprinting the details here. Suffice it to say that the results are:

Demand equations:

$X = aI/p_x$ $Y = (1-a)I/p_y$

(Note also a useful feature of a Cobb-Douglas utility function: the utility maximizing consumer spends fraction "a" of her income on x, and fraction "1-a" of her income on y.)

The calculus of utility maximization: a specific example

Consider an individual with Cobb-Douglas utility function

$U = X^{.6}Y^{(1-.6)}$

and budget constraint

$1000 = 5X + 2Y$

<u>Utility-maximizing demand equations</u>:

$X = 600/p_x$ $Y = 400/p_y$

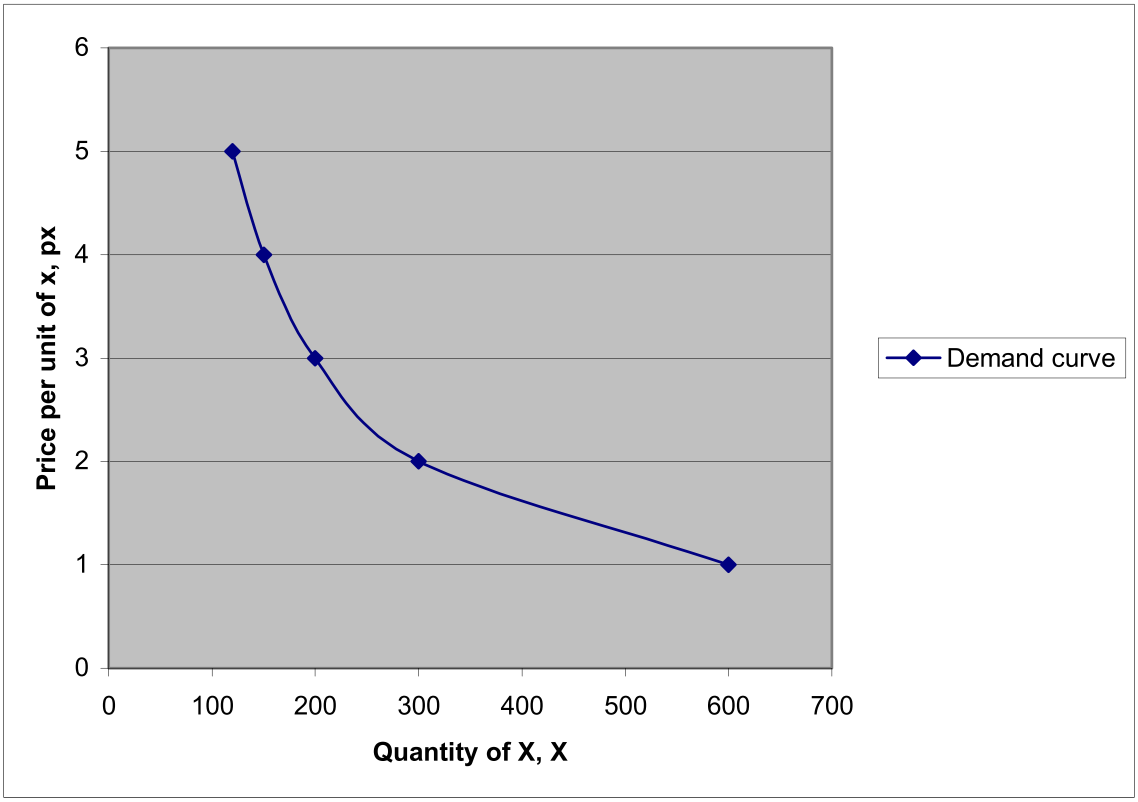
One can graph these demand equations, showing on once graph how X varies as p_x varies, and on another graph how Y varies as p_y varies. These demand curves will NOT be straight lines! Observe:

According to the demand equation $X = 600/p_x$, these points are on the individual's demand curve for x:

price per unit of x, p_x	quantity of x demanded, X
\$1	600
\$2	300
\$3	200
\$4	150
\$5	120

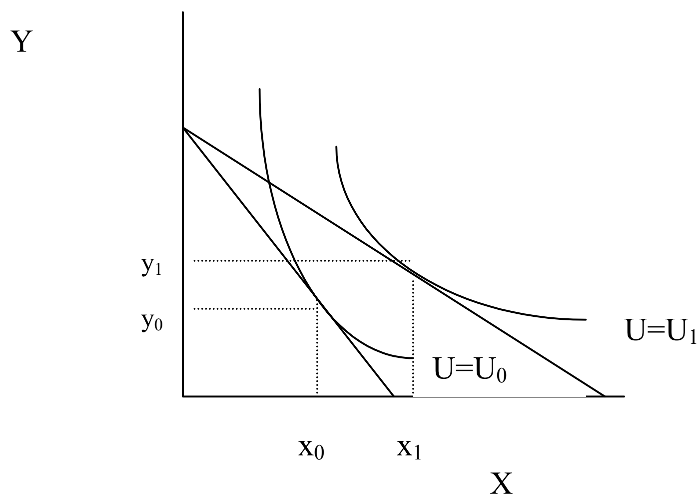
If we graph these points then we get a portion of the individual's demand curve for x on the next page:

(While I'm thinking about it, a reminder: do NOT confuse these three things: *indifference curve, budget constraint, demand curve*. They are NOT the same.)


6
5
4
3
2
1
0
0
100
200
300
400
500
600
700
Price per unit of x, px
Quantity of X, X
Demand curve

Effect of a change in price on consumption of a good

Example 1: A reduction in p_x causes the budget constraint to pivot away from the origin. The utility maximizing individual may change consumption of goods x and y. An example is below:



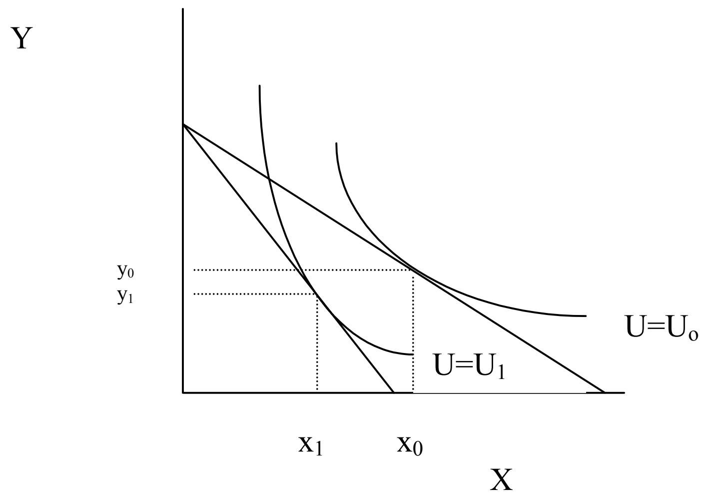
<u>Income and substitution effects:</u>
Why did the individual change consumption of good x from x_0 to x_1? The movement from x_0 to x_1 can be broken down into to effects.

substitution effect: The lower price of x means that good x is now cheaper relative to good y; one must "give up" fewer units of good y in order to consume a unit of good x. This causes one to increase demand of x.(The substitution effect is positive for a reduction in the price of x.)

income effect: The lower price of x increases the purchasing power of the individual. This causes increased consumption of "normal" goods, and reduced consumption of "inferior" goods. (Hence the income effect can be positive or negative for a reduction in the price of x.)

So: movement from x_0 to x_1 = income effect + substitution effect

Example 2: An increase in p_x causes the budget constraint to pivot toward the origin. The utility maximizing individual may change consumption of good x and y. An example is below:



Income and substitution effects:

Why did the individual change consumption of good x from x_0 to x_1? The movement from x_0 to x_1 can be broken down into to effects.

substitution effect: The higher price of x means that good x is now more expensive relative to good y; one must "give up" more units of good y in order to consume a unit of good x. (The substitution effect is negative for an increase in the price of x.)

income effect: The higher price of x reduces the purchasing power of the individual. This causes reduced consumption of "normal" goods, and increased consumption of "inferior" goods. (Hence the income effect can be positive or negative for an increase in the price of x.)

So: movement from x_0 to x_1 = income effect + substitution effect