Revenue and Profit-Maximization

Contents:

Introduction: We have recently examined *costs* quite closely, and developed the concept of *marginal cost*. In this set of notes, we shall examine *revenues* quite closely, develop the concept of *marginal revenue*, and then develop a profit maximizing rule which equates marginal revenue and marginal cost.

Important Assumptions: Throughout the analysis presented in these notes, we shall assume that the firm (a) produces only 1 product, Q, and (b) must charge the same price, P, for each unit of the product sold. Hence, we have a 1 product firm with no price discrimination.

REVENUE: A CLOSER LOOK

How does a firm's revenue change as its level of output, Q, changes?

There are 3 common ways to examine revenue:

Total revenue (TR): Just sum up the firm's total revenue from all sales. TR=PQ

Average Revenue (AR): Divide total revenue by number of units sold

AR = TR/Q

Marginal Revenue (MR): The change in total revenue when Q rises 1 unit.

More formally, using calculus: $MR = \frac{\partial TR}{\partial Q}$

Let's do 2 revenue examples below. The first example is for a firm which must lower its product's price to sell more product. The second example is for a firm which can sell more product without lowering its product's price.

Example 1: Tables

Suppose we have a firm which has this demand curve for its product:

Firm's demand curve: Q = 6 – P

Note that this firm must reduce its price in order to sell more product, as follows:

P (price per unit sold)	**Q (# of units sold)**
6	0
5	1
4	2
3	3
2	4
1	5

Let's add a column to our table for total revenue

P	**Q**	**TR (=PQ)**
6	0	$0
5	1	$5
4	2	$8
3	3	$9
2	4	$8
1	5	$5

Now let's put average revenue in our table:

P	**Q**	**TR (=PQ)**	**AR (=TR/Q)**
6	0	$0	N.A.
5	1	$5	$5
4	2	$8	$4
3	3	$9	$3
2	4	$8	$2
1	5	$5	$1

Hey! Average revenue = price! Coincidence? No! For a firm that charges the same price per unit for all units sold, average revenue—revenue per unit sold—must equal P. Know what this means? The demand curve is the average revenue curve (since the demand curve simply shows how price varies with quantity).

Average revenue = price
The firm's demand curve is its average revenue curve

Now let's put marginal revenue in our table. Remember, marginal revenue tells us the change in total revenue as Q rises an additional unit:

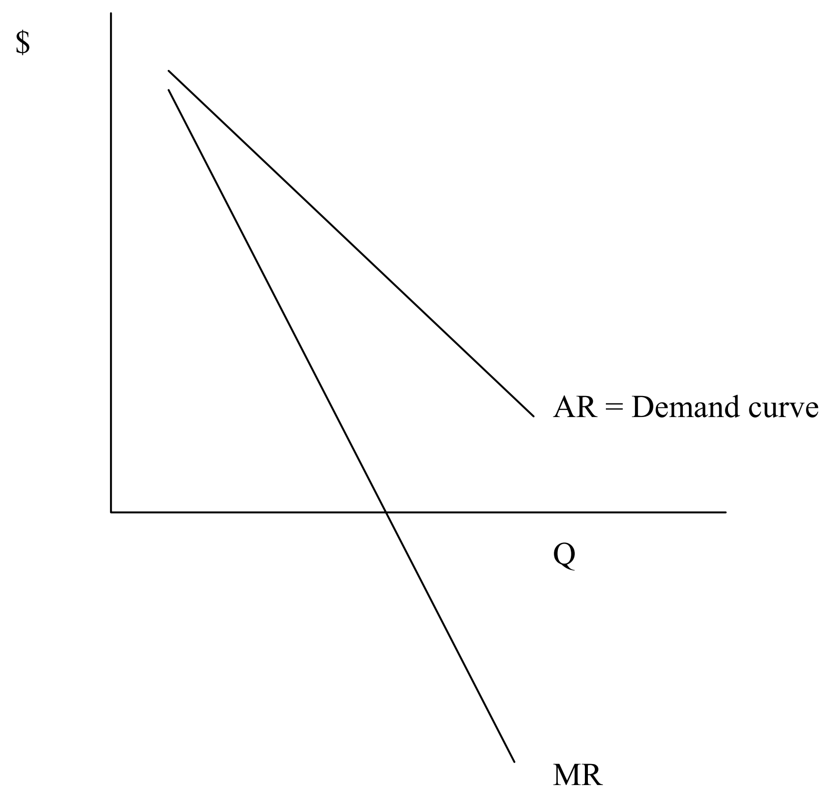
P	Q	TR (=PQ)	AR (=TR/Q)	MR
6	0	$0	N.A.	N.A.
5	1	$5	$5	$5
4	2	$8	$4	$3
3	3	$9	$3	$1
2	4	$8	$2	$-1
1	5	$5	$1	$-3

Hey! Marginal revenue becomes negative if the firm's output is larger than Q=3. Implication: this firm would NEVER want to produce more than Q=3. If, for example, it went and stupidly expanded from Q=3 to Q=4, then its total revenue would fall! Very stupid.

Note: Marginal revenue indicates the BENEFIT to the firm of producing one more unit of output, since it measures the additional dollars that this expansion generates. (Later in these notes, when we develop a profit-maximizing rule, we will see that the firm should compare this benefit of expansion to the cost of expansion—marginal cost—when deciding whether or not to expand.)

Example 1 continued: graphs

Let's graph AR and MR for our firm, using the data in the table above. (We could also graph TR, but I don't want to. Try it at home if you must.) We graph AR by plotting the points in the 2nd and 4th columns of the table. We graph MR by plotting the points in the 2nd and 5th columns of the table.



Notice that the AR curve is the same as the firm's demand curve. This makes sense, since AR=P.

Notice that the MR curve is below the AR curve. (Indeed, the MR curve is twice as steep as the AR curve.) Recalling the relationship between "marginal" and "average" things[1], it makes sense that MR is below AR. Indeed, it's the MR curve that is pulling the AR curve down (just as a student who gets progressively worse test scores as the semester progresses pulls his average down).

Example 1 continued: algebra and calculus

Let's derive equations for average revenue, total revenue and marginal revenue for our firm which has the demand curve $Q = 6 - P$

Average revenue equation: recall that AR=P, so if we rearrange the demand equation so that P is by itself on the left side of the "=", we will have an average revenue equation:

$P = 6 - Q$ or

(i) $AR = 6 - Q$

Total revenue equation: recall that TR = PQ. Since we also know that AR = P, we can substitute equation (i) into our total revenue equation, as follows:

$TR = PQ$

$R = (6-Q)Q$

(ii) $TR = 6Q - Q^2$

Marginal revenue equation: recall that $MR = \frac{\partial TR}{\partial Q}$. So we can differentiate equation (ii) with respect to Q to get a MR equation:

$$MR = \frac{\partial TR}{\partial Q}$$

$$MR = 6 - 2Q$$

We have finished our example of the revenue structure of a firm that must lower its product's price in order to sell more product. Now let's move on to example 2—a firm which does not have to lower its product's price to sell more product.

[1] Discussed in detail in the notes file "mana-costs."

Example 2: Tables

Suppose we have a firm that has this demand curve for its product:

Firm's demand curve: P = 6

What kind of wacky demand curve is this, you ask? This is the demand curve for a firm that can sell any amount of output at a constant price of $6. You see, if a firm is very small in the overall marketplace, then the level of its sales does not affect the price that it can get for each unit of its output. Consider the small tomato farmer, with only a couple of dozen acres of farmland. It sells its output in the huge U.S. tomato market, where the selling price of tomatoes is established. This tomato farm is so small that its level of production will not affect the selling price of tomatoes in the U.S.; hence this farm can sell any amount of tomatoes at the U.S. market price of tomatoes.

Note that this firm need not reduce its price in order to sell more product, as follows:

P	**Q**
6	0
6	1
6	2
6	3
6	4
6	5

Let's add a column to our table for total revenue

P	**Q**	**TR (=PQ)**
6	0	$0
6	1	$6
6	2	$12
6	3	$18
6	4	$24
6	5	$30

Now let's put average revenue in our table:

P	**Q**	**TR (=PQ)**	**AR (=TR/Q)**
6	0	$0	N.A.
6	1	$6	$6
6	2	$12	$6
6	3	$18	$6
6	4	$24	$6
6	5	$30	$6

Hey! Average revenue = price! Coincidence? No! For a firm that charges the same price per unit for all units sold, average revenue—revenue per unit sold—must equal P. Know what this means? The demand curve is the average revenue curve (since the demand curve simply shows how price varies with quantity).

Average revenue = price
The firm's demand curve is its average revenue curve

Now let's put marginal revenue in our table. Remember, marginal revenue tells us the change in total revenue as Q rises an additional unit:

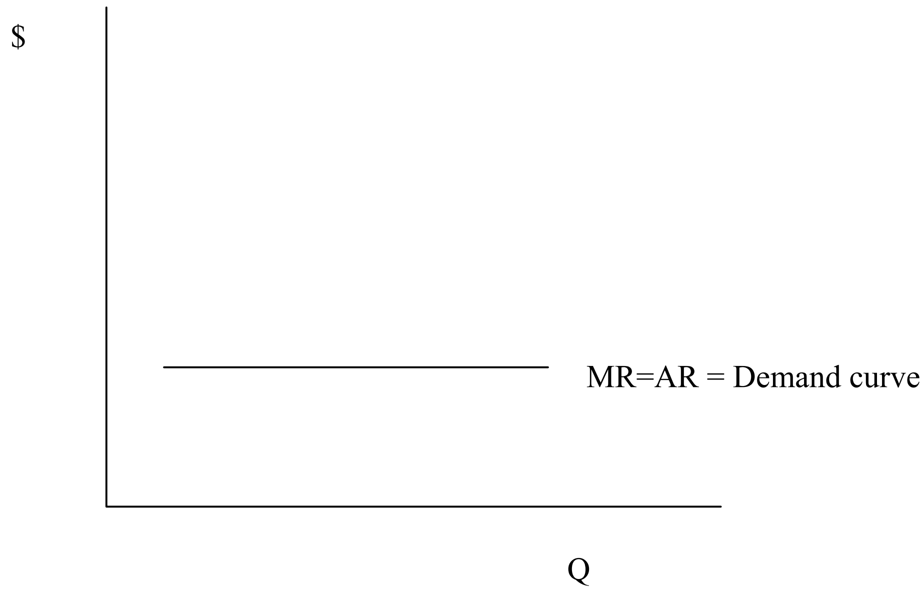
P	**Q**	**TR (=PQ)**	**AR (=TR/Q)**	**MR**
6	0	$0	N.A.	N.A.
6	1	$6	$6	$6
6	2	$12	$6	$6
6	3	$18	$6	$6
6	4	$24	$6	$6
6	5	$30	$6	$6

Hey! Marginal revenue equals average revenue!!

Note: Marginal revenue indicates the BENEFIT to the firm of producing one more unit of output, since it measures the additional dollars that this expansion generates. (Later in these notes, when we develop a profit-maximizing rule, we will see that the firm should compare this benefit of expansion to the cost of expansion—marginal cost—when deciding whether or not to expand.)

Example 2 continued: graphs

Let's graph AR and MR for our firm, using the data in the table above. (We could also graph TR, but I don't want to. Try it at the spa if you want.) We graph AR by plotting the points in the 2nd and 4th columns of the table. We graph MR by plotting the points in the 2nd and 5th columns of the table.



Notice that the AR curve is the same as the firm's demand curve. This makes sense, since AR=P.

Notice that the MR curve is on top of the AR curve. Recalling the relationship between "marginal" and "average" things, it makes sense that MR is on top of AR. Indeed, it's the MR curve that keeping the AR curve horizontal (just as a student who keeps getting the same test scores as the semester progresses keeps his average constant).

Example 2 continued: algebra and calculus

Let's derive equations for average revenue, total revenue and marginal revenue for our firm which has the demand curve $P = \$6$

Average revenue equation: recall that AR=P,

$P = 6$ or

(i) $AR = 6$

Total revenue equation: recall that TR = PQ. Since we also know that AR = P, we can substitute equation (i) into our total revenue equation, as follows:

$TR = PQ$

TR = (6)Q

(ii) TR = 6Q

Marginal revenue equation: recall that $MR = \frac{\partial TR}{\partial Q}$. So we can differentiate equation (ii) with respect to Q to get a MR equation:

$$MR = \frac{\partial TR}{\partial Q}$$

MR = 6

Note that AR and MR are constant and equal, AR = MR = 6. This is the same as our table (thankfully).

PROFIT MAXIMIZATION: MARGINAL REVENUE = MARGINAL COST

Let us develop a rule that tells the firm at which level of production it is maximizing total profits. (Recall that total profits = total revenue – total cost)

(Important note: Our rule cannot indicate to a firm whether it should shut down completely and produce zero output. It can only tell it the most profitable non-zero level of output)

Intuition behind the rule: A firm should measure the benefits of expanding output against the costs. The benefit to increased output is the additional revenue (if any) that is generated; this is marginal revenue! The cost of increased output is the additional costs that are incurred; this is marginal cost!

A firm should increase output if marginal revenue exceeds marginal cost. It should not increase output if marginal cost exceeds marginal revenue.

Now, the mathematics of the rule: Example 3
Let's use an example of a hypothetical firm with the following data on how marginal revenue and marginal cost vary with its level of output:

Q	**Marginal revenue**	**Marginal cost**	
1	100	1	
2	90	3	
3	80	13	
4	70	27	
5	60	59.99	
6	50	69	
7	30	78	
8	10	99	
9	-5	120	

What is the profit-maximizing level of Q? Let's add a 4th column—"incremental profit." This will tell the firm how much extra profit (profit *in addition to* profit already gained) the firm can obtain by increasing output by 1 more unit.

Q	**Marginal revenue**	**Marginal cost**	**Incremental profit**
1	100	1	99
2	90	3	87
3	80	13	67
4	70	27	43
5	60	59.99	.01
6	50	69	-19
7	30	78	-48
8	10	99	-89
9	-5	120	-115

Based on the above data, the profit-maximizing level of output is 5 units of output—Q=5. (If the firm only produce Q=4, it would miss out on the additional .01 of profit by expanding to Q=5. On the other hand, if the firm got too big and expanded to Q=6, it would see its profits fall by 19.)

In a mathematical world in which sellers can sell little fractions of output (e.g. 5.01 units), perhaps you can see that the profit-maximizing quantity is where MR=MC

An operating firm maximizes profits (or minimizes losses) by producing the level of output where MR =MC

MR = MC Using Calculus: Example 4

Suppose a hypothetical firm has demand curve Q = 100 – P and cost curve $C = 9Q^2$
What is this firm's profit-maximizing level of output?

Answer: First, we need equations for MR and MC. Then we need to set the equations equal and solve for Q.

The marginal cost equation:

Recall that $MC = \frac{\partial C}{\partial Q}$

Using our firm's cost equation $C = 9Q^2$

MC = 18Q

Now we need an MR equation. Our strategy:

--use the demand curve to get a total revenue equation
--differentiate the total revenue equation to get a marginal revenue equation

The demand curve is Q = 100 – P. Rearrange to get P on the left:

P = 100 – Q

Recall that total revenue = PQ. Substitute information from the demand equation:

TR = PQ

TR = (100 – Q)Q

$TR = 100Q – Q^2$

Now that we have a total revenue equation, differentiate to get MR:

MR = 100 – 2Q

Now set MR = MC and solve for Q

100 – 2Q = 18Q

20Q = 100

Q = 5

So the profit-maximizing level of output is 5.

Bonus! Let's calculate the firm's total profit:

Total revenue = $100Q – Q^2 = 100(5) – 5^2$ = \$475

Total cost = $9Q^2 = 9(5^2)$ = \$225

Total profit = total revenue – total cost = \$475 –225 = \$250

End of example.

Profit Maximization Issues

Issues:
Implicit Costs
Shut down rule

Introduction:
We have developed a profit maximizing rule: Any operating firm will maximize profits (or minimize losses) by producing the level of output where marginal revenue equals marginal cost. Now, we will discuss two issues relating to profit maximization. The first issue is to properly measure costs and profit; this relates to *implicit costs*. The second issue is to develop a shut down rule—a rule which can tell a firm in the short run whether shutting down (rather than producing where MR=MC) is its best option.

When is a firm truly profitable? Don't forget *implicit costs*
Suppose Zippy runs a business. On January 1, he puts \$1,000,000 of his money into his business, and on December 31 his business generates revenue of \$1,000,001.
Zippy's accountant would report a \$1 profit to the IRS:

Accounting profit = total revenue – total explicit costs
= 1,000,001 – 1,000,000 = \$1

But was this a good use of Zippy's money? No! He could have bought a 1 year 5% CD and earned \$50,000 interest. This lost interest is called an *implicit cost*. Economists include implicit costs in the total profit calculation:

(Economic) profit = total revenue – total cost, both explicit & implicit
= \$1,000,001 – (\$1,000,000 + \$50,000)
= -\$49,999

Interpretation: *An implicit cost is the value of a factor in its best use outside of the firm*. When implicit costs are included in the profit calculation, the firm generates a profit only if it can use its factors better than anywhere they could be used outside the firm.

Financial accountants cannot include implicit costs in their accounting for tax purposes, since the IRS does not recognize them as a cost of doing business. But they truly are, and that's why implicit costs are always included by economists in the cost (and profit) equation. (Managerial accountants often calculate implicit costs when doing internal accounting for the firm's own information.)

Note: A firm that earns a (economic) profit of $0 is sometimes said to be earning a *normal rate of return*; that is, they are, roughly speaking, doing just as well as if they had kept their money in interest-bearing assets, rather than spent it running their firm.

And now for something completely different: a **shut down rule**

<u>When should a firm shut down? *Long run* case</u>
In the long run, when all costs are variable, a firm should shut down if (economic) profit is negative (even 1 penny negative). Why? Well, they could earn a normal rate of return—zero profits--just keeping their money in interest-bearing assets. So why do worse than that by having a negative profit?

<u>When should a firm shut down? *Short run* case</u>
Suppose a firm is in the middle of a project that has already incurred some costs but has yet to generate revenue. The firm is deciding whether or not to continue the project. Here are 2 examples below. (All examples include implicit costs, and all $ amounts are expressed in present value.)

Example 1: A 12-month project began 8 months ago, and has so far incurred costs of $100,000. The firm is trying to decide now whether to continue. They forecast that continuing will incur additional costs of $40,000 and bring in additional revenue of $50,000.

→This firm should keep operating. The 1 year project will have a loss for the year, but the 1-year loss is SMALLER if they keep operating than if they stopped the project now. See?

If keep operating for rest of year:

Total profit = total revenue – (fixed cost + variable cost)
= $50,000 – ($100,000 + 40,000)
= -$90,000

If shut down (stop operating) now:

Total profit = total revenue – (fixed cost + variable cost)
= $0 – ($100,000 + $0)
= -$100,000

Example 2: A 12-month project began 8 months ago, and has so far incurred costs of $100,000. The firm is trying to decide now whether to continue. They forecast that continuing will incur additional costs of $90,000 and bring in additional revenue of $89,000.

→This firm should shut down now. The 1 year project will have a loss for the year, but the 1-year loss is LARGER if they keep operating than if they stopped the project now. See?

If keep operating for rest of year:

Total profit = total revenue – (fixed cost + variable cost)
= $89,000 – ($100,000 + 90,000)
= -$101,000

If shut down (stop operating) now:

Total profit = total revenue – (fixed cost + variable cost)
= $0 – ($100,000 + $0)
= -$100,000

<u>Comparing the two examples</u>: First, note that the $100,000 fixed costs already incurred are irrelevant in the decision to keep operating or to shut down. What matters is if the revenue can cover the variable costs. If the revenue is greater than the variable costs, then the firm will lose less money on the project if it continues to operate rather than shut down. On the other hand, alas! If the revenue can't even cover the variable costs, then the firm is better off shutting down now.

We can use this information to develop a shut down rule for a firm

Shut down if revenue < variable costs

Recall that revenue = price x quantity, P x Q
Recall also that variable costs = average variable cost x quantity, AVC x Q

Substitute:

Shut down if P x Q < AVC x Q

Cancel out the Qs:

Shut down rule:

Shut down if P < AVC

Let's combine the rule with our MR=MC rule, to have a full strategy for the firm to follow:

Profit-Maximizing Strategy

A firm will maximize profits by producing the level of Q where MR=MC, unless P<AVC at that Q. In that case, the firm should shut down (produce Q=0).

End of these notes!